On the Unity Row Summation and Real Valued Nature of the F_{LG} Matrix

Ioannis K. Dassios[1,2], Paul Cuffe[2], Andrew Keane[2]

[1]MACSI, Department of Mathematics & Statistics, University of Limerick, Ireland

[2]Electricity Research Centre, University College Dublin, Ireland

Abstract. Electrical power system calculations rely heavily on the Y_{bus} matrix, which is the Laplacian matrix of the network under study, weighted by the complex-valued admittance of each branch. It is often useful to partition the Y_{bus} into four submatrices, to separately quantify the connectivity between and among the load and generation nodes in the network. Simple manipulation of these submatrices gives the F_{LG} matrix, which offers useful insights on how voltage deviations propagate through a power system and on how energy losses may be minimized. Various authors have observed that in practice the elements of F_{LG} are real-valued and its rows sum close to one: the present paper explains and proves these properties.

Keywords: Laplacian matrix; power flow; admittance matrix

1 Introduction

Currents (I) and voltages (V) in an electrical power system are related by the admittance matrix, Y_{bus}, which is generally constructed to have the properties of a weighted Laplacian matrix (disregarding shunt element modelling), see [1], [2]. It can usefully be partitioned as follows by reordering to group generator (G) and load (L) nodes separately:

$$\begin{bmatrix} I_G \\ I_L \end{bmatrix} = Y_{bus} \begin{bmatrix} V_G \\ V_L \end{bmatrix}. \tag{1}$$

Where $I_G, V_G \in \mathbb{C}^m$, $I_L, V_L \in \mathbb{C}^n$ and

$$Y_{bus} = \begin{bmatrix} Y_{GG} & Y_{GL} \\ Y_{LG} & Y_{LL} \end{bmatrix}.$$

Where $Y_{GG} \in \mathbb{C}^{m \times m}$, $Y_{LL} \in \mathbb{C}^{n \times n}$ and $Y_{LG} = Y_{GL}^T \in \mathbb{C}^{n \times m}$. With $()^T$ we denote the (non-conjugate) transposed tensor. The Y_{bus} matrix is square by definition. It has N rows and N columns, where N is the number of nodes in the power system. The system is reordered and partitioned to group G buses and L buses separately. The n L buses plus the m G buses equals the total numbers of buses N. Although there may be buses in a power system that don't connect any L or G but are just passive interconnection points, in this analysis they would be grouped with the load buses. Typically Y_{LL} will be larger. For instance, on the $IEEE$ 118 bus system we have 19 G nodes and 99 L nodes, i.e. $Y_{GG} \in \mathbb{C}^{19 \times 19}$ and $Y_{LL} \in \mathbb{C}^{99 \times 99}$. The other sub-matrices are not generally square,

in this case $Y_{LG} \in \mathbb{C}^{99\times19}$ and $Y_{GL} \in \mathbb{C}^{19\times99}$. Note that $Y_{LG} = Y_{GL}^T$. The justification for this assertion is the guaranteed symmetry of the Y_{bus} matrix (it is constructed to have this property, which is only broken under the rare circumstance that phase-shifting transformers are included in the model). The Y_{bus} matrix is complex-valued. The physical properties of electrical conductors imply that the imaginary part of each element is larger than the real part in nearly all cases. In the below figure is plotted the real and imaginary component of every element of the Y_{bus} matrix for the 118 bus system. The imaginary component is on the vertical axis: The real:imaginary ratio is approximately equal for all
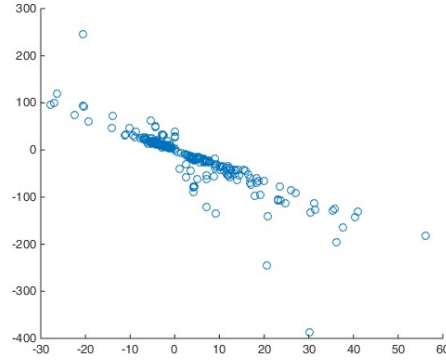

Figure 1: Empirical Real-Imaginary ratio

elements, as can be seen in Figure 1.

The partitioning of the Y_{bus} matrix into load and generation blocks was introduced by Kessel and Glavitsch in [3], and has subsequently been applied to a wide range of power engineering problems [4], [5], [6], [7], [8], [9], [10], [11], [12]. Recent work by Sikiru et. al. [13], [14], [15] has used the partitioned Y_{bus} approach to develop a more fundamental understanding of a power systems inherent connective structure, using, for instance, Schur complements and eigen analysis. Relatedly, recent work by Abdelkader et al. [16], [17], [18] offers a deeper conceptual understanding of these Y_{bus} partitions, demonstrating how they allow the separation of currents in the network into load and generator induced components. Notably [18] demonstrates how a strictly equal real:imaginary ratio for every element in the Y_{bus} matrix brings one component of the network physical power losses to zero.

From (1):

$$I_G = Y_{GG}V_G + Y_{GL}V_L \tag{2}$$

and

$$I_L = Y_{LG}V_G + Y_{LL}V_L. \tag{3}$$

Rearranging (3):

$$V_L = Z_{LL}I_L - Z_{LL}Y_{LG}V_G. \tag{4}$$

2

Where

$$Z_{LL} = \left\{ \begin{array}{lll} Y_{LL}^{-1}, & if & detY_{LL} \neq 0 \\ Y_{LL}^{\dagger}, & if & det(Y_{LL}) = 0 \end{array} \right\}.$$

The matrix $Y_{LL}^{\dagger}$ is the Moore-Penrose Pseudoinverse of Y_{LL}, calculated by the singular value decomposition of Y_{LL}, see [19], [20]. Substituting for V_L in (2):

$$I_G = (Y_{GG} - Y_{GL}Z_{LL}Y_{LG})V_G + Y_{GL}Z_{LL}I_L. \tag{5}$$

Equations (4) and (5) are typically represented in matrix form, which permits useful engineering applications:

$$\left[\begin{array}{c} V_L \\ I_G \end{array} \right] = \left[\begin{array}{cc} Z_{LL} & F_{LG} \\ K_{GL} & Y_{GGM} \end{array} \right] \left[\begin{array}{c} I_L \\ V_G \end{array} \right].$$

Where:

$$Y_{GGM} = Y_{GG} - Y_{GL}Z_{LL}Y_{LG}$$

and

$$F_{LG} = -Z_{LL}Y_{LG} = -K_{GL}^{T} \tag{6}$$

Various works (e.g [14], [4], [21]) have noted, and, indeed, relied upon, the observation that the elements of F_{LG} are in practice real-valued and that its rows sum to unity, or close to.

2   Main results

As written in the previous section, the ratio $(Im : Re)$ of each entry of the Y_{bus} matrix will in practice tend to be fairly homogeneous though this isn't always guaranteed. The Y_{bus} matrix will have generally diagonal elements that are positive real and negative imaginary. Off-diagonals will be negative real and positive imaginary. Based on these observations we can provide the following Proposition.

Proposition 2.1. Assume the Y_{bus} matrix as defined in (1). If the entries in each row have the real:imaginary ratio equal, then all entries of the F_{LG} matrix, defined in (6), are real numbers. In addition if the Y_{bus} matrix has diagonal elements that are non-negative real, non-positive imaginary and off diagonals are non-positive real, non-negative imaginary, then all entries of F_{LG}, are non-negative real numbers.

Proof. Let

$$Y_{LG} = \left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{array} \right], \quad Y_{LL} = \left[\begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{array} \right]$$

3

and $a_{kj} = Re(a_{kj}) + iIm(a_{kj})$, $b_{kj} = Re(b_{kj}) + iIm(b_{kj})$. Then for each row $k = 1, 2, ..., n$ of Y_{LG}, Y_{LL} we have

$$\frac{Im(a_{kj})}{Re(a_{kj})} = \frac{Im(b_{kj})}{Re(b_{kj})} = u_k \in \mathbb{R}, \quad k = 1, 2, ..., n.$$

From (6) we have

$$Y_{LL}F_{LG} = -Y_{LG},$$

or, equivalently,

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} F_{LG} = - \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Re(b_{11})[1 + iu_1] & Re(b_{12})[1 + iu_1] & \dots & Re(b_{1n})[1 + iu_1] \\ Re(b_{21})[1 + iu_2] & Re(b_{22})[1 + iu_2] & \dots & Re(b_{2n})[1 + iu_2] \\ \vdots & \vdots & \ddots & \vdots \\ Re(b_{n1})[1 + iu_n] & Re(b_{n2})[1 + iu_n] & \dots & Re(b_{nn})[1 + iu_n] \end{bmatrix} F_{LG} =$$

$$- \begin{bmatrix} Re(a_{11})[1 + iu_1] & Re(a_{12})[1 + iu_1] & \dots & Re(a_{1m})[1 + iu_1] \\ Re(a_{21})[1 + iu_2] & Re(a_{22})[1 + iu_2] & \dots & Re(a_{2m})[1 + iu_2] \\ \vdots & \vdots & \ddots & \vdots \\ Re(a_{n1})[1 + iu_n] & Re(a_{n2})[1 + iu_n] & \dots & Re(a_{nm})[1 + iu_n] \end{bmatrix},$$

or, equivalently,

$$diag\left\{ 1 + iu_1, \quad 1 + iu_2, \quad \dots, \quad 1 + iu_n \right\} \begin{bmatrix} Re(b_{11}) & Re(b_{12}) & \dots & Re(b_{1n}) \\ Re(b_{21}) & Re(b_{22}) & \dots & Re(b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ Re(b_{n1}) & Re(b_{n2}) & \dots & Re(b_{nn}) \end{bmatrix} F_{LG} =$$

$$-diag\left\{ 1 + iu_1, \quad 1 + iu_2, \quad \dots, \quad 1 + iu_n \right\} \begin{bmatrix} Re(a_{11}) & Re(a_{12}) & \dots & Re(a_{1m}) \\ Re(a_{21}) & Re(a_{22}) & \dots & Re(a_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ Re(a_{n1}) & Re(a_{n2}) & \dots & Re(a_{nm}) \end{bmatrix}.$$

or, equivalently, since $det(diag\left\{ 1 + iu_1, \quad 1 + iu_2, \quad \dots, \quad 1 + iu_n \right\}) \neq 0$

$$Re(Y_{LL})F_{LG} = -Re(Y_{LG}).$$

This is a liner system where the known matrices have only real entries. Hence, any solution for F_{LG} lies inside of $\mathbb{R}^{n \times m}$, i.e. all entries of F_{LG} are real numbers. In addition since

the Y_{bus} matrix is assumed to have diagonal elements that are non-negative real & non-positive imaginary and off diagonals are non-positive real & non-negative imaginary, the matrix $Re(Y_{LG})$ has all its entries non-positive, i.e. the matrix $Re(Y_{LG})$ has all its entries non-negative and the matrix $Re(Y_{LL})$ has its diagonal elements non-negative and its off diagonals non-positive, i.e. the inverse (or pseudo inverse) of $Re(Y_{LL})$ has all its entries non-negative, see [22]. Hence the matrix F_{LG} will have all its elements non-negative since it is given from the product of two matrices, the inverse (or pseudo inverse) of $Re(Y_{LL})$ and the matrix $Re(Y_{LG})$. The proof is completed.

If we use a short line model of the system and so neglect shunt elements, then each row of the Y_{bus} matrix sums to zero. If we include shunts, then each row sums close to zero. We can state the following Theorem.

Theorem 2.1. Assume the Y_{bus} matrix as defined in (1) and the matrix F_{LG}, as defined in (6). Then the row sum of F_{LG} is one if the rows of Y_{bus} sum to zero and $det(Y_{LL}) \neq 0$ and is close to one if the rows of Y_{bus} sum close to zero or sum to zero and $det(Y_{LL}) = 0$.

Proof. Let for system (1)

$$Y_{LG} = [a_{ij}]_{i=1,2,\ldots,n}^{j=1,2,\ldots,m} \tag{7}$$

and

$$Y_{LL} = [b_{ij}]_{i=1,2,\ldots,n}^{j=1,2,\ldots,n}. \tag{8}$$

Where

$$[a_{ij}]_{i=1,2,\ldots,n}^{j=1,2,\ldots,m} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}, \quad [b_{ij}]_{i=1,2,\ldots,n}^{j=1,2,\ldots,n} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix}.$$

We assume that each row of Y_{bus} sums to zero, i.e. $\forall$ row $i = 1, 2, \ldots, n$ of the matrices (7), (8) we have $\sum_{j=1}^{m} a_{ij} + \sum_{j=1}^{n} b_{ij} = 0$, or equivalently

$$\sum_{j=1}^{m} a_{ij} = -\sum_{j=1}^{n} b_{ij}, \quad \forall i = 1, 2, \ldots, n. \tag{9}$$

From (6) we have

$$Y_{LL} F_{LG} = -Y_{LG}, \tag{10}$$

Let

$$F_{LG} = [c_{ij}]_{i=1,2,\ldots,n}^{j=1,2,\ldots,m}. \tag{11}$$

Then $\forall i = 1, 2, \ldots, n$ by substituting (7), (8), (11) into (10) and by calculating every component in row i, we get

$$-a_{i1} = \sum_{k=1}^{n} b_{ik} c_{k1}$$
$$-a_{i2} = \sum_{k=1}^{n} b_{ik} c_{k2}$$
$$\vdots$$
$$-a_{im} = \sum_{k=1}^{n} b_{ik} c_{km}.$$

5

By taking the sum of the above equalities, $\forall i = 1, 2, ..., n$ we arrive at

$$-[a_{i1} + a_{i2} + .. + a_{im}] = \sum_{k=1}^{n} b_{ik}c_{k1} + \sum_{k=1}^{n} b_{ik}c_{k2} + \cdots + \sum_{k=1}^{n} b_{ik}c_{km},$$

or, equivalently,

$$-\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{k=1}^{n} b_{ik}c_{kj}.$$

By using (9) on the above expression we get

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{m}\sum_{k=1}^{n} b_{ik}c_{kj},$$

or, equivalently,

$$\sum_{k=1}^{n} b_{ik} = \sum_{j=1}^{m}\sum_{k=1}^{n} b_{ik}c_{kj},$$

or, equivalently,

$$b_{i1} + b_{i2} + \cdots + b_{in} = \sum_{j=1}^{m}[b_{i1}c_{1j} + b_{i2}c_{2j} + \cdots + b_{in}c_{nj}],$$

or equivalently,

$$b_{i1} + b_{i2} + \cdots + b_{in} = \quad b_{i1}c_{11} + b_{i2}c_{21} + \cdots + b_{in}c_{n1} + $$
$$b_{i1}c_{12} + b_{i2}c_{22} + \cdots + b_{in}c_{n2} + $$
$$\vdots$$
$$b_{i1}c_{1m} + b_{i2}c_{2m} + \cdots + b_{in}c_{nm},$$

or, equivalently,

$$b_{i1} + b_{i2} + \cdots + b_{in} = \quad b_{i1}(c_{11} + c_{12} + \cdots + c_{1m)} + $$
$$b_{i2}(c_{21} + c_{22} + \cdots + c_{2m}) + $$
$$\vdots$$
$$b_{in}(c_{n1} + c_{n2} + \cdots + c_{nm}),$$

or, equivalently,

$$b_{i1} + b_{i2} + \cdots + b_{in} = b_{i1}\left(\sum_{j=1}^{m} c_{1j}\right) + b_{i2}\left(\sum_{j=1}^{m} c_{2j}\right) + \cdots + b_{in}\left(\sum_{j=1}^{m} c_{nj}\right),$$

or, equivalently,

$$b_{i1}\left[\left(\sum_{j=1}^{m} c_{1j}\right) - 1\right] + b_{i2}\left[\left(\sum_{j=1}^{m} c_{2j}\right) - 1\right] + \cdots + b_{in}\left[\left(\sum_{j=1}^{m} c_{nj}\right) - 1\right] = 0. \tag{12}$$

6

Let $\forall i = 1, 2, ..., n$

$$q_i = (\sum_{j=1}^{m} c_{ij}) - 1. \tag{13}$$

Then by replacing (13) into (12), $\forall i = 1, 2, ..., n$ we have

$$b_{i1}q_1 + b_{i2}q_2 + \cdots + b_{in}q_n = 0,$$

or, equivalently,

$$\begin{bmatrix} b_{i1} & b_{i2} & \dots & b_{in} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = 0, \quad \forall i = 1, 2, ..., n.$$

Since the above expression holds $\forall i = 1, 2, ..., n$ we have equivalently

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By setting $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$, $0_{n,1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and by using (8) we have

$$Y_{LL} \cdot q = 0_{n,1}.$$

From the above expression if $det(Y_{LL}) \neq 0$, then $q = 0_{n,1}$, or, equivalently, $q_i = 0$, $\forall i = 1, 2, ..., n$, i.e. by using (13)

$$\sum_{j=1}^{m} c_{ij} = 1. \tag{14}$$

If $det(Y_{LL}) = 0$, then $q \cong 0_{n,1}$ (by using the pseudo inverse of Y_{LL} via the SVD method), or, equivalently, $q_i \cong 0$, $\forall i = 1, 2, ..., n$, i.e. by using (13)

$$\sum_{j=1}^{m} c_{ij} \cong 1. \tag{15}$$

Thus, from (14) and (15) every row of F_{LG} will sum to 1 if $det(Y_{LL}) \neq 0$ and will sum closely to 1 if $det(Y_{LL}) = 0$. If the rows of Y_{bus} sum close to zero, then with similar steps we arrive at (15). The proof is completed.

Conclusions

The partitioning of the Y_{bus} matrix has opened numerous fruitful avenues in power system analysis, and it is hoped that a clearer understanding of the matrix properties underpinning these partitions will support this strand of research. This work has proved two numerically observed properties of the F_{LG} matrix: future work may consider the matrix characteristics of the other sub-matrices derived by the Y_{bus} partitioning approach.

Acknowledgments

References

[1] R. J. Sanchez-Garcia, M. Fennelly, S. Norris, N. Wright, G. Niblo, J. Brodzki, and J. W. Bialek, *Hierarchical Spectral Clustering of Power Grids,* Power Systems, IEEE Transactions on, vol. 29, pp. 2229-2237, 2014.

[2] R. Merris, *Laplacian matrices of graphs: a survey,* Linear Algebra and its Applications, vol. 197?198, pp. 143-176, 1994.

[3] P. Kessel and H. Glavitsch, *Estimating the Voltage Stability of a Power System,* Power Delivery, IEEE Transactions on, vol. 1, pp. 346-354, 1986.

[4] G. Yesuratnam and D. Thukaram, *Congestion management in open access based on relative electrical distances using voltage stability criteria,* Electric Power Systems Research, vol. 77, pp. 1608-1618, 2007.

[5] K. Visakha, D. Thukaram, and L. Jenkins, *Transmission charges of power contracts based on relative electrical distances in open access,* Electric Power Systems Research, vol. 70, pp. 153-161, 2004.

[6] D. Thukaram and C. Vyjayanthi, *Relative electrical distance concept for evaluation of network reactive power and loss contributions in a deregulated system,* Generation, Transmission & Distribution, IET, vol. 3, pp. 1000-1019, 2009.

[7] C. Vyjayanthi and D. Thukaram, *Evaluation and improvement of generators reactive power margins in interconnected power systems,* Generation, Transmission & Distribution, IET, vol. 5, pp. 504-518, 2011.

[8] S. Aboreshaid and R. Billinton, *Probabilistic evaluation of voltage stability,* Power Systems, IEEE Transactions on, vol. 14, pp. 342-348, 1999.

[9] S. Surendra and D. Thukaram, *Identification of prospective locations for generation expansion with least augmentation of network,*" Generation, Transmission & Distribution, IET, vol. 7, pp. 37-45, 2013.

[10] R. Billinton and S. Aboreshaid, *Voltage stability considerations in composite power system reliability evaluation,* Power Systems, IEEE Transactions on, vol. 13, pp. 655-660, 1998.

[11] J. Hongjie, Y. Xiaodan, and Y. Yixin, *An improved voltage stability index and its application,* International Journal of Electrical Power & Energy Systems, vol. 27, pp. 567-574, 2005.

[12] G. M. Huang and N. C. Nair, *Voltage stability constrained load curtailment procedure to evaluate power system reliability measures,* in Power Engineering Society Winter Meeting, 2002. IEEE, 2002, pp. 761-765 vol.2.

[13] T. H. Sikiru, A. A. Jimoh, Y. Hamam, J. T. Agee, and R. Ceschi, *Classification of networks based on inherent structural characteristics,* in Transmission and Distribution: Latin America Conference and Exposition (T&D-LA), 2012 Sixth IEEE/PES, 2012, pp. 1-6.

[14] T. H. Sikiru, A. A. Jimoh, and J. T. Agee, *Inherent structural characteristic indices of power system networks,* International Journal of Electrical Power & Energy Systems, vol. 47, pp. 218-224, 2013.

[15] T. H. Sikiru, A. A. Jimoh, Y. Hamam, J. T. Agee, and R. Ceschi, *Voltage profile improvement based on network structural characteristics,* in 6th IEEE transmission and distribution latin America conference, Montevideo, Uruguay, 2012.

[16] S. M. Abdelkader, D. J. Morrow, and A. J. Conejo, *Network usage determination using a transformer analogy,* Generation, Transmission & Distribution, IET, vol. 8, pp. 81-90, 2014.

[17] S. M. Abdelkader and D. Flynn, *A new method for transmission loss allocation considering the circulating currents between generators,* European Transactions on Electrical Power, vol. 20, pp. 1177-1189, 2010.

[18] S. M. Abdelkader, *Characterization of Transmission Losses,* Power Systems, IEEE Transactions on, vol. 26, pp. 392-400, 2011.

[19] B. Datta, *Numerical linear algebra and applications,* Siam, (2010).

[20] G. Golub, C.V. Loan, *Matrix computations.* Vol. 3. JHU Press, (2012).

[21] G. Yesuratnam and M. Pushpa, *Congestion management for security oriented power system operation using generation rescheduling,* in Probabilistic Methods Applied to Power Systems (PMAPS), 2010 IEEE 11th International Conference on, 2010, pp. 287-292.

[22] R.F. Gantmacher, *The theory of matrices I, II*, Chelsea, New York, (1959).